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Remarks on the complete integrability of dynamical systems with fermionic variables

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Abstract. We study the role of $(1,1)$ graded tensor field T in the analysis of complete integrability of dynamical systems with fermionic variables. We find that such a tensor T can be a recursion operator if and only if T is even as a graded map, namely, if and only if $p(T) = 0$. We clarify this fact by constructing an odd tensor for two examples, a supersymmetric Toda chain and a supersymmetric harmonic oscillator. We explicitly show that it cannot be a recursion operator since it does not allow new constants of motion to be built from the first two, in contrast to what usually happens with ordinary, i.e. non-graded systems.

1. Introduction

In recent years there has been renewed interest in completely integrable Hamiltonian systems, specially in connection with the study of integrable quantum field theory, Yang–Baxter algebras and, more recently, quantum groups.

Integrability criteria available both in finite and infinite dimensions have been established by methods directly related to group theory and to familiar procedures in classical mechanics [1, 2], by looking at soliton equations as dynamical systems on (infinite-dimensional) phase manifold [8, 9, 12, 13, 18, 26]. This last approach was also suggested by the occurrence in the inverse scattering transform of a peculiar operator, the so-called recursion operator [16], which naturally fits in this geometrical setting as a mixed tensor field on the phase manifold. This tensor has to satisfy several requirements, the most important being that its Nijenhuis torsion [11, 22] vanishes.

There have been several attempts to analyse integrability of fermionic dynamical systems (see for instance [6, 15, 19]), and to extend to such systems [17], in an algorithimic sense at least, results and techniques used for bosonic dynamics based on the role of recursion operators. In particular, one would like to define a graded Nijenhuis torsion.

In this paper we address these issues. We show that a mixed $(1,1)$ graded tensor field T can act as a recursion operator if and only if T is an even map.

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There are dynamical systems, like supersymmetric Witten's dynamics [28] which allow a bi-Hamiltonian description with an even and odd Hamiltonian function and in terms of an even and an odd Poisson structure respectively (so that the dynamical vector field is always even) [25, 27]. This allows construction of an odd tensor field which could be a good candidate as a recursion operator. We explicitly show that this is not possible.

The paper is organized as follows. First, we define notation and recall the formulation of complete integrability in terms of a $(1,1)$ tensor field available in the bosonic case. After a résumé of graded differential calculus and graded Poisson structures, we analyse a supersymmetric harmonic oscillator and a supersymmetric Toda chain (both are examples of Witten's supersymmetric dynamics). We then prove that for an odd $(1,1)$ tensor, a $(2,1)$ tensor corresponding to its torsion (graded-Nijenhuis torsion) cannot be defined and that a graded $(1,1)$ tensor cannot be a recursion operator unless it is even. Finally, we present some conclusions.

2. Complete integrability and recursion operators in the bosonic case

Complete integrability of Hamiltonian systems with finitely many degrees of freedom is exhaustively characterized by the Liouville–Arnold theorem [1, 2]. Here we briefly recall an alternative characterization in terms of an invariant (under the dynamical evolution) $(1,1)$ tensor field T. Examples of such tensors can also be constructed for systems with infinitely many degrees of freedom, so that the approach described could be of use in the latter cases as well.

We shall deal only with smooth, i.e. C^∞ objects, and notation will follow as closely as possible that of [1] and [20]. In particular if M is a (finite-dimensional) ordinary manifold we denote by $\mathcal{F}(M)$ the ring of real valued functions on M, by $\mathcal{X}(M)$ the Lie algebra of vector fields, by $\mathcal{X}(M)^*$ its dual of forms and by $\mathcal{T}_1^1(M)$ the mixed $(1,1)$ tensor fields.

Associated with every $T \in \mathcal{T}_1^1(M)$ there are two endomorphisms of $\mathcal{X}(M)$ and $\mathcal{X}(M)^*$ which are defined by

$$\hat{T} : \mathcal{X}(M) \longrightarrow \mathcal{X}(M) \qquad \check{T} : \mathcal{X}(M)^* \longrightarrow \mathcal{X}(M)^*$$

$$T(X,\alpha) =: \hat{T}X \rfloor \alpha =: X \rfloor \check{T}\alpha \qquad \forall\, X \in \mathcal{X}(M) \qquad \alpha \in \mathcal{X}(M)^* . \tag{1}$$

The *Nijenhuis tensor* (or *torsion*) of T is the $(2,1)$ tensor field $\mathbf{N}_T$ defined by [11, 22]

$$\mathbf{N}_T(X,Y;\alpha) =: \mathbf{H}_T(X,Y) \rfloor \alpha \tag{2}$$

where $\mathbf{H}_T : \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M)$ is the $\mathcal{F}(M)$-linear map given by

$$\mathbf{H}_T(X,Y) =: \hat{T}^2[X,Y] + [\hat{T}X, \hat{T}Y] - \hat{T}[\hat{T}X, Y] - \hat{T}[X, \hat{T}Y] . \tag{3}$$

Equivalently, this can be written as

$$\mathbf{H}_T(X,Y) = [\widehat{L_{\hat{T}X}T} - \hat{T} \circ \widehat{L_X T}](Y) \qquad \forall\, X, Y \in \mathcal{X}(M) . \tag{4}$$

To simplify our notation, in the following, when no confusion arises, we shall denote both the endomorphisms $\hat{T}$ and $\check{T}$ with the same symbol, namely T.

From (2) it is clear that the vanishing of the tensor $\mathbf{N}_T$ is equivalent to the vanishing of $\mathbf{H}_T$, namely $\mathbf{N}_T \equiv 0$ iff $\mathbf{H}_T \equiv 0$.

The following proposition has been proved in [9]:

Proposition 1. A dynamical vector field Γ which admits a mixed tensor field T, which is invariant ($L_\Gamma T = 0$), with vanishing Nijenhuis torsion, diagonalizable with doubly degenerate eigenvalues λ, without stationary points ($\mathrm{d}\lambda \neq 0$) is separable integrable and Hamiltonian, i.e. is a separable completely integrable Hamiltonian system.

The proof is given observing that: $\mathrm{N}_T = 0$ implies the integrability, in the Frobenius sense, of the eigenspaces of T; $L_\Gamma T = 0$ implies the separability of Γ along the eigenmanifolds in dynamics with one degree of freedom, each of which has a constant of motion.

A $(1,1)$ tensor field with the previously stated properties, acts as a 'recursion operator' [12, 18], i.e. when iteratively applied to Γ one produces symmetries $\Gamma_k = \hat{T}^k\Gamma$ or constants of motion H_k by $\mathrm{d}H_k = \check{T}^k \mathrm{d}H$.

The main property of the tensor field T in the analysis of complete integrability of its infinitesimal automorphisms is the vanishing of its Nijenhuis tensor $\mathrm{N}_T = 0$. It is then plausible that a suitable generalization of such a condition could play an important role in analysing the integrability of dynamical systems with fermionic degrees of freedom. Moreover, it seems natural to think that such a generalization could come from a graded generalization of some of the following relations which are available in the bosonic case:

(a) $\mathrm{N}_T = 0 \Longrightarrow \mathrm{Im}\, T$ is a Lie algebra.
(b) $\mathrm{N}_T = 0$, $\mathrm{d}(T\mathrm{d}H) = 0 \Longrightarrow \mathrm{d}(T^k\mathrm{d}H) = 0$.
(c) $\mathrm{N}_T = 0 \Longleftrightarrow \mathrm{d}_T \circ \mathrm{d}_T = 0$; here d_T is a suitable generalization of the exterior derivative associated with any $(1,1)$ tensor field [21].
(d) $T =: \Lambda_1^{-1} \circ \Lambda_2$, $\mathrm{N}_T = 0 \Longleftrightarrow \Lambda_1 + \Lambda_2$ satisfies the Jacobi identity. Here Λ_1 and Λ_2 are two Poisson structures.
(e) $\omega(X,Y) =: [TX,Y] + [X,TY] - T[X,Y]$; $T\omega(X,Y) = [TX,TY]$ (this is the same as $\mathrm{N}_T = 0$) $\Longleftrightarrow [X,Y]_\lambda =: [X,Y] + \lambda\omega(X,Y)$ satisfies the Jacobi identity for any value of the real parameter λ.

One could expect that some, if not all, of the previous relations do not hold true in the graded situation.

Before we proceed with the analysis of the graded Nijenhuis condition we shall give a brief review of the graded differential calculus on supermanifolds which will be followed by the study of some simple examples.

3. Graded differential calculus

We review some fundamentals of supermanifold theory [10, 23] while referring to the literature for a mathematically coherent definition [3, 24]. In the following, by graded we shall always mean $\mathbf{Z}_2$-graded.

The basic algebraic object is a real exterior algebra $B_L = (B_L)_0 \oplus (B_L)_1$ with L generators. An (m,n)-dimensional supermanifold is a topological manifold S modelled over the 'vector superspace'

$$B_L^{m,n} = (B_L)_0^m \times (B_L)_1^n \tag{5}$$

by means of an atlas whose transition functions fulfil a suitable 'supersmoothness' condition. A supersmooth function $f : U \subset B_L^{m,n} \to B_L$ has the usual superfield expansion

$$f(x^1 \dots x^m,\ \theta^1 \dots \theta^n) = f_0(x) + \sum_{\alpha=1}^{n} f_\alpha(x)\,\theta^\alpha + \cdots + f_{1\dots n}(x)\,\theta^1 \cdots \theta^n \qquad (6)$$

where the xs are the even (Grassmann) coordinates, the θs are the odd ones, and the dependence of the coefficient functions $f_{\dots}(x)$ on the even variables is fixed by their values for real arguments.

We shall denote by $\mathcal{G}(S)$ and $\mathcal{G}(U)$ the graded ring of supersmooth B_L-valued functions on S and $U \subset S$, respectively.

The class of supermanifolds which, up to now, turns out to be relevant for applications in physics is given by the De Witt supermanifolds. They are defined in terms of a coarse topology on $B_L^{m,n}$, called the De Witt topology, whose open sets are the counterimages of open sets in $\mathbf{R}^m$ through the body map $\sigma^{m,n} : B_L^{m,n} \to \mathbf{R}^m$. An (m,n) supermanifold is De Witt if it has an atlas such that the images of the coordinate maps are open in the De Witt topology. A De Witt (m,n) supermanifold is a locally trivial fibre bundle over an ordinary m-manifold S_0 (called the body of S) with a vector fibre [23]. It is these not surprising that, modulo some technicalities, a De Witt supermanifold can be identified with a Berezin–Konstant supermanifold [4, 14].

The graded tangent space TS is constructed in the following manner. For each $x \in S$, let $\mathcal{G}(x)$ be the germs of functions at x, and denote by T_xS the space of graded B_L-linear maps $X : \mathcal{G}(x) \to B_L$ which satisfy the Leibnitz rule. Then, T_xS is a free graded B_L-module of dimension (m,n), and the disjoint union $\bigcup_{x\in S} T_xS$ can be given the structure of a rank (m,n) super vector bundle over S, denoted by TS. The sections $\mathcal{X}(S)$ of TS are a graded $\mathcal{G}(S)$-module and are identified with the graded Lie algebra $\mathrm{Der}\,\mathcal{G}(S)$ of derivations of $\mathcal{G}(S)$. Derivations (or vector fields) are said to be even (or odd) if they are even (or odd) as maps (satisfying in addition a graded Leibnitz rule) from $\mathcal{G}(S) \to \mathcal{G}(S)$. A local basis is given by

$$\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^m}, \frac{\partial}{\partial \theta^1}, \dots, \frac{\partial}{\partial \theta^n}. \qquad (7)$$

Remark. Unless explicitly stated, by using a partial derivative we shall always mean a left derivative, namely a derivative acting from the left. In general, if $z^i = (x^j, \theta^k)$, when acting on any homogeneous function $f \in \mathcal{G}(S)$, left and right derivatives are related by

$$\frac{\overrightarrow{\partial}}{\partial z^i} f = (-1)^{p(z^i)[p(f)+1]} f \frac{\overleftarrow{\partial}}{\partial z^i} \qquad i \in \{1, \dots, m+n\}. \qquad (8)$$

In a similar way one defines the cotangent space and bundle. $T_x^\star S$ is the space of graded B_L-linear maps from $T_x(S) \to B_L$ and $T^\star S = \bigcup_{x\in S} T_x^\star S$. $T_x^\star S$ is a free graded B_L-module of dimension (m,n), while $T^\star S$ is a rank (m,n) super vector bundle over S. The sections $\mathcal{X}(S)^*$ of $T^\star S$ are a graded $\mathcal{G}(S)$-module and are identified with the graded $\mathcal{G}(S)$-linear maps from $\mathrm{Der}\,\mathcal{G}(S) \to \mathcal{G}(S)$. They are the 1-forms on S. Forms are said to be even (or odd) if they are even (or odd) as maps $\mathcal{X}(M) \to \mathcal{G}(S)$.

In general, a p covariant and q contravariant graded tensor is any graded $\mathcal{G}(S)$-multilinear map $\alpha : \mathcal{X}(S) \times \cdots \times \mathcal{X}(S) \times \mathcal{X}(S)^* \times \cdots \times \mathcal{X}(S)^* \longrightarrow \mathcal{G}(S)$ (p $\mathcal{X}(S)$

factors and q $\mathcal{X}(S)^*$ factors). The collection of all rank (p,q) tensors is a graded $\mathcal{G}(S)$-module.

A graded p-form is a skew-symmetric covariant graded tensor of rank p. We denote by $\Omega^p(S)$ the collection of all p-forms. The *exterior differential* on S is defined by letting $X \rfloor \mathrm{d} f = X(f)\ \forall f \in \mathcal{G}(S),\ X \in \mathcal{X}(S)$ and is extended to maps $\Omega^p(S) \to \Omega^{p+1}(S),\ p \geqslant 0$, in the usual way, so that $\mathrm{d}^2 = 0$. If $X_i \in \mathcal{X}(S)$ are homogeneous elements,

$$X_1 \wedge \ldots \wedge X_{p+1} \rfloor \mathrm{d}\varphi =: \sum_{i=1}^{p+1} (-1)^{a(i)} X_i (X_1 \wedge \ldots \overset{i}{\checkmark} \ldots \wedge X_{p+1} \rfloor \varphi) + \sum_{1 \leqslant i < j \leqslant p} (-1)^{b(i,j)} [X_i, X_j] \wedge X_1 \wedge \ldots \overset{i}{\checkmark} \ldots\ldots \overset{j}{\checkmark} \ldots \wedge X_{p+1} \rfloor \varphi \tag{9}$$

where

$$\begin{aligned} a(i) &= 1 + i + p(X_i) \sum_{h=1}^{i-1} p(X_h) \\ b(i,j) &= i + j + p(X_i) \sum_{h=1}^{i-1} p(X_h) + p(X_j) \sum_{\substack{h=1 \\ h \neq i}}^{j-1} p(X_h) . \end{aligned} \tag{10}$$

By definition one has that $p(d) = 0$.

The Lie derivative $L_{(\cdot)}$ of forms is defined by

$$\begin{aligned} &L_{(\cdot)} : \mathcal{X}(S) \times \Omega^p(S) \to \Omega^p(S) \\ &L_X = X \rfloor \circ \mathrm{d} + \mathrm{d} \circ X \rfloor \qquad \forall X \in \mathcal{X}(S) . \end{aligned} \tag{11}$$

Clearly, $p(L_X) = p(X)$.

The Lie derivative of any tensor product can be defined in an obvious manner by requiring the Leibnitz rule and can be extended to any tensor by using linearity.

Suppose now that we have a tensor $T \in T^1_1(M)$ which is homogeneous of degree $p(T)$. Again we can define two graded endomorphisms of $\mathcal{X}(S)$ and $\mathcal{X}(S)^*$ by the formulae (in the following two formulae X, Y are homogeneous elements in $\mathcal{X}(S)$ while α is any element in $\mathcal{X}(S)^*$)

$$\begin{aligned} &\hat{T} : \mathcal{X}(S) \longrightarrow \mathcal{X}(S) \qquad \check{T} : \mathcal{X}(S)^* \longrightarrow \mathcal{X}(S)^* \\ &T(X, \alpha) =: \hat{T} X \rfloor \alpha =: (-1)^{p(X)p(T)} X \rfloor \check{T} \alpha . \end{aligned} \tag{12}$$

We could be tempted to define a graded Nijenhuis torsion of T by a relation analogous to (2)

$$\begin{aligned} {}^G\mathbf{N}_T(X, Y; \alpha) &=: {}^G\mathbf{H}_T(X, Y) \rfloor \alpha \\ {}^G\mathbf{H}_T(X, Y) &=: \hat{T}^2 [X, Y] + (-1)^{p(T)p(X)} [\hat{T}X, \hat{T}Y] - \hat{T}[\hat{T}X, Y] \\ &\quad - (-1)^{p(T)p(X)} \hat{T}[X, \hat{T}Y] . \end{aligned} \tag{13}$$

Proposition 2. The map ${}^G\mathbf{H}_T : \mathcal{X}(S) \times \mathcal{X}(S) \to \mathcal{X}(S)$ defined in (13) is $\mathcal{G}(S)$-linear and graded antisymmetric if and only if $p(T) = 0$.

Proof. Just compute.

Remark. When $p(T) = 1$, the map defined in (13) is not antisymmetric or linear over even functions, nor when it is restricted to even vector fields. Therefore (12) and (13) define a graded tensor (which is in addition graded antisymmetric) if and only if $p(T) = 0$.

4. Poisson supermanifold

We briefly describe how to introduce super Poisson structures on an (m, n)-dimensional supermanifold S [4, 17]. For additional results see also [5]. As before, we shall denote by $z^i = (x^j, \theta^k)$, $i \in \{1, \ldots, m+n\}$ the local coordinates on S. The following proposition is in [4] and can be proved by direct calculations.

Proposition 3. Let $\|\omega^{ij}\|$ be a $(m+n)\times(m+n)$ matrix (depending upon the point $z \in S$) with the following properties:

(1) the elements ω^{ij} are homogeneous with parity $p(\omega^{ij}) = p(z^i)+p(z^j)+p(\omega)$ and $p(\omega)$ not depending on the indices i and j;

(2)

$$\omega^{ji} = -(-1)^{[p(z^i)+p(\omega)][p(z^j)+p(\omega)]}\omega^{ij} \tag{14}$$

(3)

$$(-1)^{[p(z^i)+p(\omega)][p(z^l)+p(\omega)]}\omega^{is}\frac{\vec{\partial}}{\partial z^s}\omega^{jl}+(-1)^{[p(z^l)+p(\omega)][p(z^j)+p(\omega)]}\omega^{ls}\frac{\vec{\partial}}{\partial z^s}\omega^{ij} + (-1)^{[p(z^j)+p(\omega)][p(z^i)+p(\omega)]}\omega^{js}\frac{\vec{\partial}}{\partial z^s}\omega^{li} = 0\,. \tag{15}$$

Then, the following bracket

$$\{F, G\} =: F\frac{\overleftarrow{\partial}}{\partial z^i}\omega^{ij}\frac{\vec{\partial}}{\partial z^j}G \tag{16}$$

makes $\mathcal{G}(S)$ a Lie superalgebra (Poisson superstructure).

We have two different kinds of structure: For $p(\omega) = 0$, an even Poisson structure; for $p(\omega) = 1$ an odd Poisson structure. Indeed, one can check that the bracket (16) has properties

$$\{F, G\} = -(-1)^{[p(F)+p(\omega)][p(G)+p(\omega)]}\{G, F\} \tag{17}$$

$$(-1)^{[p(F)+p(\omega)][p(H)+p(\omega)]}\{\{F, G\}, H\} + (-1)^{[p(G)+p(\omega)][p(F)+p(\omega)]}\{\{G, H\}, F\} + (-1)^{[p(H)+p(\omega)][p(G)+p(\omega)]}\{\{H, F\}, G\} = 0\,. \tag{18}$$

We infer from (17) and (18) that, when thought of as elements of the Poisson superalgebra, homogeneous elements of $\mathcal{G}(S)$ preserve their parity if $p(\omega) = 0$, while they change it if $p(\omega) = 1$.

If the matrix $||\omega^{ij}||$ is regular, then its inverse $||\omega_{ij}||$, $\omega_{ij}\omega^{jk} = \delta_i^k$, gives the components of a symplectic structure $\omega = \frac{1}{2}\mathrm{d}z^i \wedge \mathrm{d}z^j \omega_{ji}$, namely, ω is closed and non-degenerate with the properties

$$\begin{aligned} p(\omega_{ij}) &= p(z^i) + p(z^j) + p(\omega) \\ \omega_{ji} &= -(-1)^{p(z^i)p(z^j)}\omega_{ij} \end{aligned} \tag{19}$$

and ω is homogeneous with parity just equal to $p(\omega)$.

There is also a Darboux theorem [17]

Proposition 4. Let (S, ω) be an (m, n)-dimensional symplectic manifold with ω homogeneous. Then

(1) If $p(\omega) = 0$, then $\dim\ S = (2r, n)$ and there exist local coordinates such that

$$\omega = \mathrm{d}q^i \wedge \mathrm{d}p^i + \mathrm{d}\xi^j \wedge \mathrm{d}\xi^j \qquad \omega_{ij} = \begin{pmatrix} 0 & \mathbf{I}_r & 0 \\ -\mathbf{I}_r & 0 & \\ 0 & 0 & \mathbf{I}_n \end{pmatrix} . \tag{20}$$

(2) If $p(\omega) = 1$, then $\dim\ S = (m, m)$ and there exist local coordinates such that

$$\omega = \mathrm{d}u^i \wedge \mathrm{d}\xi^i \qquad \omega_{ij} = \begin{pmatrix} 0 & \mathbf{I}_m \\ -\mathbf{I}_m & 0 \end{pmatrix} . \tag{21}$$

Having a Poisson structure we can deal with Hamilton equations. From (16), if H is the Hamiltonian, the corresponding equations are

$$\dot{z}^i = \omega^{ij} \frac{\vec{\partial}}{\partial z^j} H . \tag{22}$$

Now we would like to maintain the possibility of explicitly constructing the flow of (22). This requires that the dynamical evolution be an even vector field. In turn this implies that the Poisson structure and the Hamiltonian function should have the same parity; in particular, with an odd Poisson structure we need an odd Hamiltonian function.

5. Examples

Before we analyse the graded Nijenhuis condition we present a few examples.

5.1. Mixed bosonic–fermionic harmonic oscillator

The mixed bosonic–fermionic harmonic oscillator in $(2,2)$ dimensions is described with coordinates (q, p, η, ξ) and has the following equations of motion

$$\dot{q} = p \qquad \dot{p} = -q \qquad \dot{\eta} = \xi \qquad \dot{\xi} = -\eta\,. \tag{23}$$

Equations (23) can be given two Hamiltonian descriptions. The Hamiltonians are: the usual even one

$$H = \tfrac{1}{2}(p^2 + q^2) + \mathrm{i}\xi\eta \tag{24}$$

and an odd one

$$K = p\xi + q\eta \tag{25}$$

while the two Poisson structures are respectively

$$\Lambda_H = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & \mathrm{i} & 0 \\ 0 & 0 & 0 & \mathrm{i} \end{pmatrix} \qquad \omega_H = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -\mathrm{i} & 0 \\ 0 & 0 & 0 & -\mathrm{i} \end{pmatrix} \tag{26}$$

and

$$\Lambda_K = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \qquad \omega_K = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \tag{27}$$

We can construct a a mixed invariant tensor field T by

$$T =: \omega_H \circ \Lambda_K = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -\mathrm{i} & 0 & 0 \\ \mathrm{i} & 0 & 0 & 0 \end{pmatrix}. \tag{28}$$

However, this odd tensor field ($p(T) = 1$) is not a recursion operator. One can easly find that

$$\begin{aligned} &T\mathrm{d}K = \mathrm{d}H \\ &T\mathrm{d}H = -\mathrm{i}(\mathrm{d}q)\xi + \mathrm{i}(\mathrm{d}p)\eta - \mathrm{i}(\mathrm{d}\eta)p + \mathrm{i}(\mathrm{d}\xi)q \qquad \mathrm{d}(T\mathrm{d}H) \neq 0\,. \end{aligned} \tag{29}$$

If we evaluate the Poisson brackets of the coordinate variables with the two symplectic structures (26) and (27) we find that

$$\{q,p\}_H = 1 \qquad \{p,q\}_H = -1 \qquad \{\eta,\eta\}_H = \mathrm{i} \qquad \{\xi,\xi\}_H = \mathrm{i} \tag{30}$$

and

$$\{q,\xi\}_K = 1 \qquad \{\xi,q\}_K = -1 \qquad \{p,\eta\}_K = -1 \qquad \{\eta,p\}_K = 1 \tag{31}$$

the remaining ones being identically zero. We see that the sum $\{\cdot,\cdot\}_+$ of the two structures is itself a Poisson structure with the property

$$\{F, G\}_+ = -(-1)^{p(F)p(G)}\{G, F\}_+ \tag{32}$$

but it has no definite parity. Moreover $\{\cdot,\cdot\}_+$ is degenerate.

5.2. *Witten dynamics [28]*

Interesting examples come from supersymmetric dynamics. It has been shown [25,27] that the dynamics of Witten's Hamiltonian systems [28] can be given a bi-Hamiltonian description with an even Poisson bracket and Grassmann-even Hamiltonian or with an odd bracket and Grassmann-odd Hamiltonians. Instead of considering the general case we shall study a supersymmetric Toda chain with coordinates (q, p, η, ξ).

The even Hamiltonian is given by

$$H = \tfrac{1}{2}(p^2 + \mathrm{e}^q) + \tfrac{1}{2}\mathrm{i}\xi\eta\mathrm{e}^{q/2}. \tag{33}$$

With the even Poisson structure

$$\Lambda_H = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & \mathrm{i} & 0 \\ 0 & 0 & 0 & \mathrm{i} \end{pmatrix} \qquad \omega_H = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -\mathrm{i} & 0 \\ 0 & 0 & 0 & -\mathrm{i} \end{pmatrix} \tag{34}$$

the equations of motion read

$$\dot q = p \qquad \dot p = -\tfrac{1}{2}\mathrm{e}^q - \tfrac{1}{4}\mathrm{i}\xi\eta\mathrm{e}^{q/2} \qquad \dot\eta = \tfrac{1}{2}\xi\mathrm{e}^{q/2} \qquad \dot\xi = -\tfrac{1}{2}\eta\mathrm{e}^{q/2}. \tag{35}$$

Then the following functions are constants of the motion:

$$K = p\xi + \mathrm{e}^{q/2}\eta \qquad L = p\eta - \mathrm{e}^{q/2}\xi \qquad F = \mathrm{i}\xi\eta. \tag{36}$$

We can use K in (36) (or L) as an alternative Hamiltonian. The corresponding symplectic structure can be written as

$$\begin{aligned}\omega_K &= \mathrm{d}q\wedge\mathrm{d}\xi + \mathrm{d}p\wedge\mathrm{d}q(\mathrm{e}^{-q/2}\eta) + \mathrm{d}p\wedge\mathrm{d}\eta(-2\mathrm{e}^{-q/2}) + \mathrm{d}f\wedge\mathrm{d}H \\ &= \mathrm{d}\{\mathrm{d}q(-\xi) + \mathrm{d}p(2\mathrm{e}^{-q/2}\eta) + f\mathrm{d}H\}\end{aligned} \tag{37}$$

where $f(q, p, \eta, \xi)$ is a function explicitly given by

$$f = A\xi + B\eta$$

$$\begin{aligned} A(q,p) &= \frac{1}{p^2+\mathrm{e}^q}\left(\frac{2p}{\sqrt{p^2+\mathrm{e}^q}}\log\left(\frac{\mathrm{e}^{q/2}}{p+\sqrt{p^2+\mathrm{e}^q}}\right) + \frac{2\mathrm{e}^{q/2}}{\sqrt{p^2+\mathrm{e}^q}} - 2\right) \\ B(q,p) &= \frac{1}{p^2+\mathrm{e}^q}\left(\frac{2\mathrm{e}^{q/2}}{\sqrt{p^2+\mathrm{e}^q}}\log\left(\frac{\mathrm{e}^{q/2}}{p+\sqrt{p^2+\mathrm{e}^q}}\right) - \frac{2p}{\sqrt{p^2+\mathrm{e}^q}} - 2p\mathrm{e}^{-q/2}\right).\end{aligned} \tag{38}$$

If Γ is the dynamical vector field of the Toda system, as given by (35), then the function f is such that $\mathrm{i}_\Gamma\mathrm{d}f = \mathrm{e}^{-q/2}\eta$ and this, in turn, assures that $\mathrm{i}_\Gamma\omega_K = \mathrm{d}K$.

It take some algebra to check that the $(1,1)$ tensor field

$$T = \omega_K \circ \Lambda_H$$

is such that

$$\begin{aligned} T\mathrm{d}H &= \mathrm{d}K \\ \mathrm{d}(T^2\mathrm{d}H) &\neq 0. \end{aligned} \tag{39}$$

Again, T in (38) is not a recursion operator.

6. Super Nijenhuis torsion

One of the most relevant consequences deriving from a (non-graded) (1-1) tensor field T with vanishing Nijenhuis torsion is the possibility of generating sequences of exact 1-forms according to

Proposition 5.

$$\mathbf{N}_T = 0 \qquad \mathrm{d}(T\mathrm{d}F) = 0 \Longrightarrow \mathrm{d}(T^k \mathrm{d}F) = 0\,. \tag{40}$$

Proof. Let α be any 1-form. By using the expression of the exterior derivative, after some algebra one finds that

$$X \wedge Y \lrcorner\, \mathrm{d}(T^2\alpha) = \{X \wedge TY + TX \wedge Y\} \lrcorner\, \mathrm{d}(T\alpha) - \{TX \wedge TY\} \lrcorner\, \mathrm{d}\alpha - \mathbf{H}_T(X, Y) \lrcorner\, \alpha \tag{41}$$

where $\mathbf{H}_T$ is defined in (3). Assume now that both α and $T\alpha$ are closed. From (41) we see that $T^2\alpha$ is closed if and only if $\mathbf{H}_T = 0$, namely if and only if the Nijenhuis torsion of T vanishes.

Let us analyse now the graded situation. Suppose T is a graded $(1,1)$ tensor field which is homogeneous of parity $p(T)$. Then, if α is any 1-form, by using definition (9), after some (graded) algebra, the analogue of (41) reads

$$\begin{aligned} X \wedge Y \lrcorner\, \mathrm{d}(T^2\alpha) = \{&(-1)^{p(T)p(Y)} X \wedge TY \\ &+ (-1)^{p(T)[p(X)+p(Y)]} TX \wedge Y\} \lrcorner\, \mathrm{d}(T\alpha) \\ &- (-1)^{p(T)[p(X)+p(T)]} TX \wedge TY \lrcorner\, \mathrm{d}\alpha \; - (-1)^{p(T)}\, {}^{G}\mathbf{H}_T(X, Y) \lrcorner\, \alpha \\ &+ (-1)^{p(T)p(X)} [1 - (-1)^{p(T)}] L_{TX}(TY \lrcorner\, \alpha) \end{aligned} \tag{42}$$

where ${}^{G}\mathbf{H}_T$ is defined in (13).

It is clear then, that for a $(1,1)$ odd tensor a $(2,1)$ tensor corresponding to its torsion (super Nijenhuis torsion) can be defined only when $p(T) = 0$. The same result is attained with the use of the general approach $\mathrm{d}_T \circ \mathrm{d}_T = 0$.

7. Conclusions

Summing up, we have shown that there are examples of dynamical systems whose dynamical vector field Γ admits two Hamiltonian descriptions, odd and even, and that the tensor field T constructed out of the corresponding Poisson structures is not a recursion operator since it cannot generate new integrals of motion after the first two.

We have also shown that this fact is general and that for a generic graded $(1,1)$ tensor field T a graded Nijenhuis torsion cannot be defined unless T is even.

From the nature of the proof it seems plausible that a similar theorem should hold true also in infinite dimensions.

The 'no go' theorem we have proved in our paper does not exhaust, obviously, the analysis of complete integrability for graded Hamiltonian systems. Much more attention must be paid, however, in generalizing to the graded case geometrical structures which play a relevent and natural role in the non-graded situation.

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